Linear Algebra

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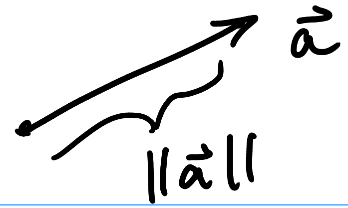
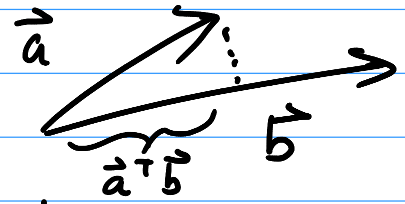
Vectors



$$\vec{x} = \begin{bmatrix}1\\0\\2\end{bmatrix} \in \mathbb{R}^3 \qquad \vec{x}^T = [1, 0, 2]$$

inner product – projection



$$\vec{a}^T\vec{b} = \underset{(1\times 3)}{[a_1, a_2, a_3]} \underset{(3\times 1)}{\begin{bmatrix} b_1\\b_2\\b_3\end{bmatrix}} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$= \vec{b}^T\vec{a} \quad \text{(only works for real)}$$

$$= \|\vec{a}\| \, \|\vec{b}\| \cos\theta$$

= dot product $\vec{a} \cdot \vec{b}$
scalar

$$\|\vec{a}\|_2 \text{ or } \|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\sum_{i=1}^{d} a_i^2}$$

ℓ_2 norm or 2-norm, Euclidean

outer product — takes 2 vectors
gives a matrix

$$\vec{a}\,\vec{b}^T = \underset{(3\times1)}{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}} \underset{(1\times3)}{\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}} = \underset{3\times3}{\begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix}}$$

$$\begin{bmatrix} a_1\,b_1 & a_1\,b_2 & a_1\,b_3 \\ a_2\,b_1 & a_2\,b_2 & a_2\,b_3 \\ a_3\,b_1 & a_3\,b_2 & a_3\,b_3 \end{bmatrix}$$

rank-1 matrix

i.e.

$$\underbrace{\|\vec{a}\|\cdot\|\vec{b}\|}_{\sigma_1}\ \underbrace{\frac{\vec{a}}{\|\vec{a}\|}}_{\vec{u}}\ \underbrace{\frac{\vec{b}^T}{\|\vec{b}\|}}_{\vec{v}^T}$$

Matrices

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \\ \begin{bmatrix} b_{21} & b_{22} & b_{23} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \end{bmatrix}$$

2×3 $\quad$ 3×1

$$= \begin{bmatrix} b_{11} a_1 + b_{12} a_2 + b_{13} a_3 \\ b_{21} a_1 + b_{22} a_2 + b_{23} a_3 \end{bmatrix}$$

multiply AB, A $(m \times n)$, B $(n \times d)$, AB $(m \times d)$

$$\text{for}_{\substack{i=1,\dots,m \\ j=1,\dots,d}} \; (AB)_{i,j} = \sum_{k=1}^{n} a_{i,k} \, b_{k,j}$$

transpose $\quad \vec{a}^T \longrightarrow$ row vector

$$[\ B\ ] \longrightarrow [\ B^T\ ]$$

$(2\times3) \qquad\qquad (3\times2)$

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \longrightarrow \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \\ b_{13} & b_{23} \end{bmatrix}$$

$$(B^T)_{ij} = (B)_{ji}$$

$$C = AB, \quad C^T = (AB)^T = B^T A^T$$

$c = \vec{a}^T\vec{b}, \quad c^T = c = \vec{b}^T\vec{a}$ $\qquad$ ☆ reverses order ☆

Inverse of a matrix

$A\vec{x} = \vec{b}$

<u>Defn</u> A is <u>singular</u> if it has no inverse. Otherwise, it's <u>non-singular</u> or <u>invertible</u>.

A^{-1} another matrix so that $A^{-1}A = \underset{(n \times n)}{I} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$

If $A \in \mathbb{R}^{n \times n}$ and all s.v.'s > 0
(i.e. nonsingular, full rank)
then $A^{-1} \in \mathbb{R}^{n \times n}$ and it exists.

$\underset{(n \times n)}{C} = \underset{(n \times 2n)}{A}\,\underset{(2n \times n)}{B}$, $\quad C^{-1} = (AB)^{-1} = B^{-1}A^{-1}$

assuming B^{-1}, A^{-1} exist

Identity matrix

$$I_n = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

$(n \times n)$

$$I\vec{x} = \vec{x}$$

$$A^{-1} A \vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = I\vec{x} = A^{-1}\vec{b}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \cdot 1 + x_2 \cdot 0 \\ x_1 \cdot 0 + x_2 \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

np.eye

Diagonal matrix

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$$

$n \times n$

$$D\vec{x} = \begin{bmatrix} d_1 x_1 \\ d_2 x_2 \\ \vdots \\ d_n x_n \end{bmatrix}$$

Symmetric matrix

A is symmetric iff. $A = A^T$ ... must be square

$a_{ij} = a_{ji}$

ex/ $\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$

ex/ $X^T X = A$ covariance

$A^T = (X^T X)^T = X^T (X^T)^T$

$= X^T X = A$

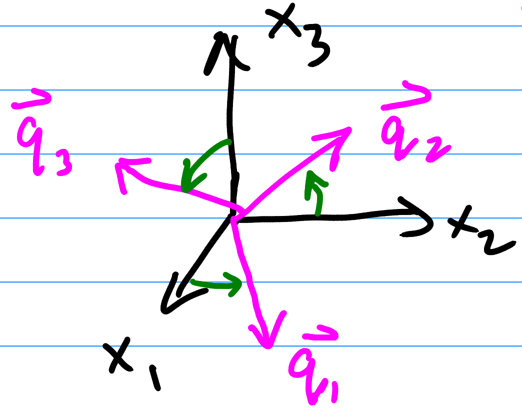
(also positive semidefinite = no negative eigenvalues)

Orthogonal matrix

Q is orthogonal iff $Q^{-1} = Q^T$.

$$\Rightarrow QQ^T = Q^TQ = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

dot products



$$\begin{bmatrix} | & | & | \\ \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \\ | & | & | \end{bmatrix}$$

q_1: $\|\vec{q}_1\| = 1 = \vec{q}_1^{\,T}\vec{q}_1$

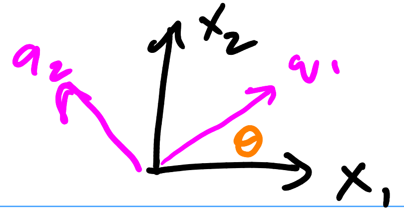
q_2: $\vec{q}_2^{\,T}\vec{q}_1 = 0$, $\|\vec{q}_2\| = 1$

q_3: $\vec{q}_3^{\,T}\vec{q}_1 = 0$ $\quad \|\vec{q}_3\| = 1$

$\vec{q}_3^{\,T}\vec{q}_2 = 0$

Gram-Schmidt orthogonalization

ex/ in 2-D



$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\vec{q}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\|\vec{q}_1\|^2 = \cos^2\theta + \sin^2\theta = 1$$

pick $\theta \in [0, 2\pi]$

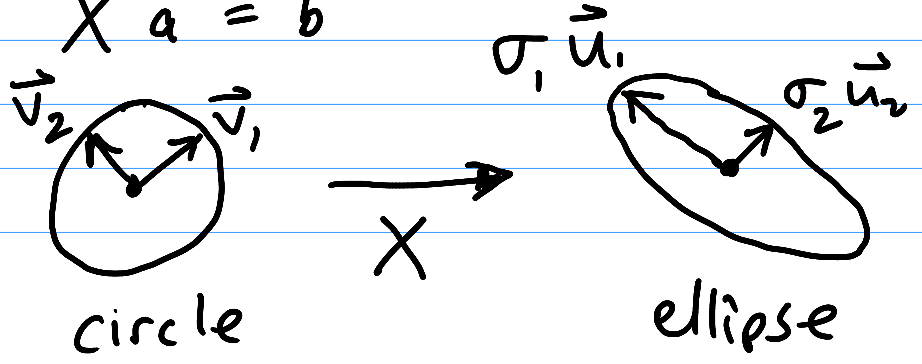
ex/ $X = U S V^T$

U, V orthogonal

if X is $n \times n$

$\vec{v}_1 \xrightarrow{X} \vec{u}_1 \sigma_1$

$X\vec{a} = \vec{b}$



"Matrix times circle equals ellipse"

Eigenvalues / eigenvectors

eigen = "characteristic" (of the matrix)

A <u>square</u> $n \times n$

$$A\vec{v} = \lambda \vec{v}$$

$\Leftrightarrow$ $\vec{v}$ is an eigenvector with eigenvalue λ

$$A v = V \Lambda V^{-1} v$$

$$V = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$$

$$AV = V\Lambda$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$\Updownarrow$

$$A\vec{v}_1 = \lambda_1 \vec{v}_1, \quad A\vec{v}_2 = \lambda_2 \vec{v}_2, \quad \ldots$$

Special case: $A = A^T \Rightarrow A = V \Lambda \underbrace{V^T}_{= V^{-1} = V^T}$