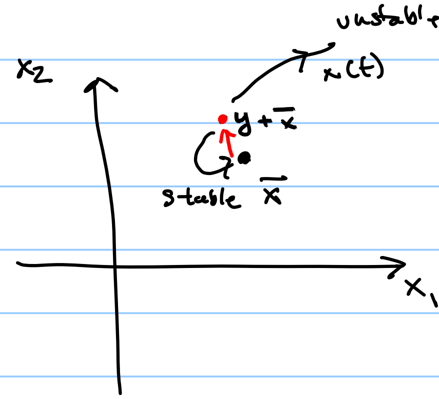
Reference: www.scholarpedia.org/article/Stability

Linearization and stability of equilibria

Illustrate in 2 dimensions:



Equilibrium $\bar{x}$: $\underset{\sim}{f}(\underset{\sim}{\bar{x}}) = 0$

CONCEPT:

1) Perturb equilibrium: $\bar{x} \rightarrow \bar{x} + y$

2) Ask: what happens to solution starting at $\bar{x}+y$? Does it RETURN to equilibrium $\bar{x}$? If so, $\bar{x}$ is stable.

OR, does solution trajectory "MOVE AWAY"? Then $\bar{x}$ is UNSTABLE.

A BIT MORE PRECISELY:

Say $\bar{x}$ is STABLE if, for y near $\bar{x}$, $x(t) \underset{t\to\infty}{\longrightarrow} \bar{x}$

UNSTABLE if $x(t)$ can escape a neighborhood $\mathcal{U}$ of $\bar{x}$, no matter how near y is to $\bar{x}$.

STRATEGY: STUDY DYNAMICS OF SMALL PERTURBATION $y(t)$ AWAY FROM EQUILIB. $\bar{x}$

$$\frac{dx}{dt} = f(x)$$

; consider trajectory $x(t) = \bar{x} + y(t)$

$y_j(t) << 1$: trajectory is near $\bar{x}$.

$$\frac{d}{dt}\left(\bar{x} + y(t)\right) = f(\bar{x}+y(t))$$

$$\underbrace{\frac{d}{dt}\bar{x}}_{0} + \frac{d}{dt}y(t) = \frac{d}{dt}y(t) \longrightarrow \boxed{\frac{d}{dt}y(t) = f(\bar{x}+y(t))}$$

Now, get simple expression for $f(\bar{x} + y(t))$

Consider small $\underset{\sim}{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$; $y_1, y_2 \ll 1$

In 2-D: $\underset{\sim}{f}(\underset{\sim}{\bar{x}} + \underset{\sim}{y}) = \begin{pmatrix} f_1(\bar{x}_1 + y_1, \bar{x}_2 + y_2) \\ f_2(\bar{x}_1 + y_1, \bar{x}_2 + y_2) \end{pmatrix} \approx$

$$\begin{pmatrix} f_1(\bar{x}_1, \bar{x}_2) \\ f_2(\bar{x}_1, \bar{x}_2) \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1}{\partial x_1} y_1 + \frac{\partial f_1}{\partial x_2} y_2 \\ \frac{\partial f_2}{\partial x_1} y_1 + \frac{\partial f_2}{\partial x_2} y_2 \end{pmatrix} = \underline{\underset{\sim}{f}(\underset{\sim}{\bar{x}}) + Df\, \underset{\sim}{y}}$$

, where

$$Df(\bar{x}_1, \bar{x}_2) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_1, x_2) & \frac{\partial f_1}{\partial x_2}(x_1, x_2) \\ \frac{\partial f_2}{\partial x_1}(x_1, x_2) & \frac{\partial f_2}{\partial x_2}(x_1, x_2) \end{bmatrix}$$

<u>JACOBIAN MATRIX</u>.

Since $f(\bar{x}) = 0$ by definition, we have:

$$\boxed{\frac{dy}{dt} = \underset{\substack{\uparrow \\ \text{Fixed matrix}}}{Df}\; \underset{\sim}{y}}$$

Solutions to this equation:
in terms of EIGENVALUES $\{\lambda_1, \lambda_2\}$ and EIGENVECTORS of $Df(\bar{x})$

FACT: if $Df(\bar{x})$ has two different evl's λ_1, λ_2, write solution to (i) as:

$$y(t) = c_1 e^{\lambda_1 t} \underset{\sim}{v_1} + c_2 e^{\lambda_2 t} \underset{\sim}{v_2}$$

To demonstrate this, just plug:

$$\dot{y} = c_1 \exp(\lambda_1 t)\, \lambda_1 \underset{\sim}{v_1} + c_2 \exp(\lambda_2 t)\, \lambda_2 \underset{\sim}{v}$$

$$= c_1 \exp(\lambda_1 t)\, Df(\bar{x})\, \underset{\sim}{v_1} + c_2 \exp(\lambda_2 t)\, Df(\bar{x})\, \underset{\sim}{v_2}$$

$$= Df(\underset{\sim}{x})\, y \qquad \checkmark$$

. . .

CONCLUDE: (see Sec 5.3.3 of E+G) that

$\bar{x}$ is stable if $\lambda_1 < 0$ and $\lambda_2 < 0$

$\bar{x}$ is UNSTABLE if $\lambda_1 > 0$ or $\lambda_2 > 0$

Unclear if $\lambda_1 = 0$ or $\lambda_2 = 0$.

$$\frac{dx_1}{dt} = -x_1 + x_1^2 = f_1(x_1, x_2)$$

$$\frac{dx_2}{dt} = -x_2 + x_2^2 = f_2(x_1, x_2)$$

$$\bar{x} = (0, 0)$$

$$J(0,0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} (0,0) =$$

$$= \begin{pmatrix} -1+2x_1 & 0 \\ 0 & -1+2x_1 \end{pmatrix} (0,0) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Eigenvalues $= (-1, -1) \longrightarrow$ <u>Stable</u>