Leveraging the lacuna: Spectral gaps and tensor recovery

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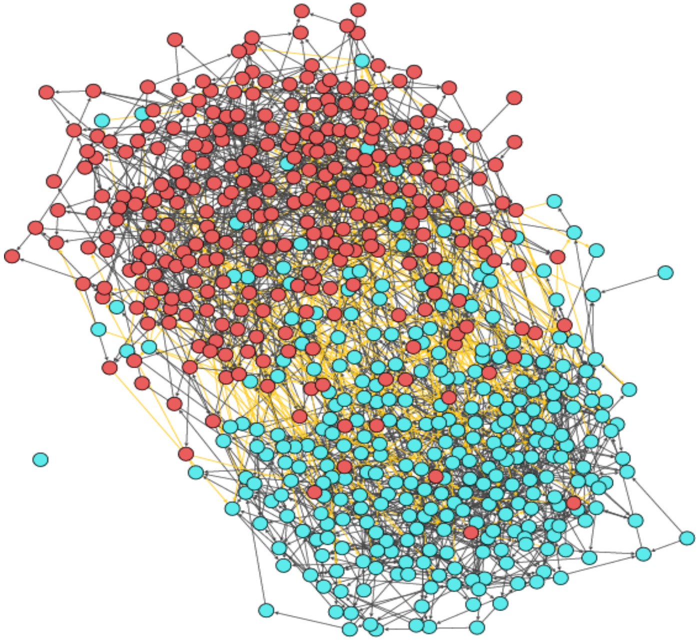
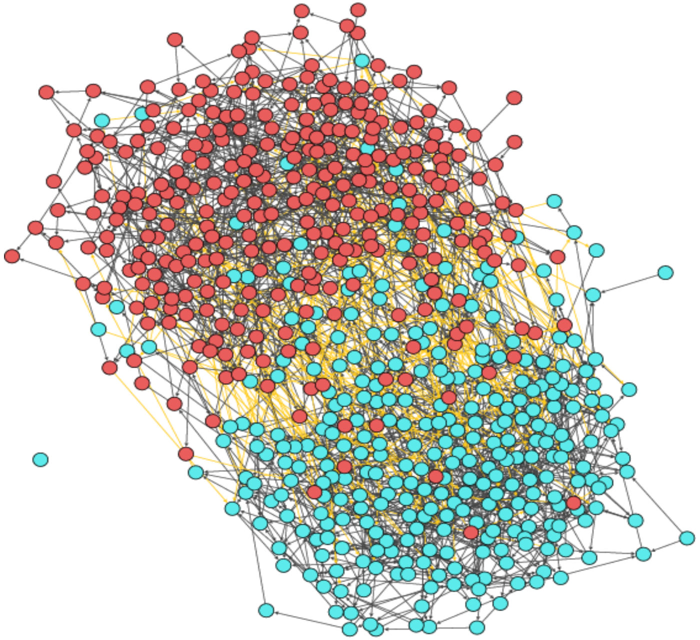
Joint work with Yizhe Zhu, UCSD Math

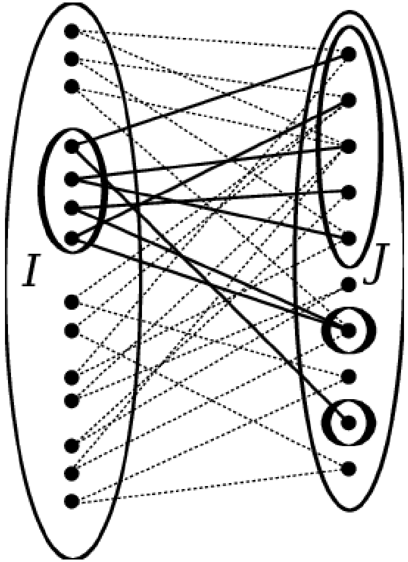


Joshua Mendoza, Liu et al (2012)



Expander graphs

Expander graphs



- Sparse yet highly connected
  - Satisfy strong isoperimetric inequalities
  - Every vertex set has many neighbors
  - Every cut has many edges crossing
  - Random walks converge quickly to stationary

Informal definitions from Reingold, Vadhan, Wigderson (2000)

Review: Hoory, Linial, Widgerson (2006)

de Castro (2010)

Spectral expansion

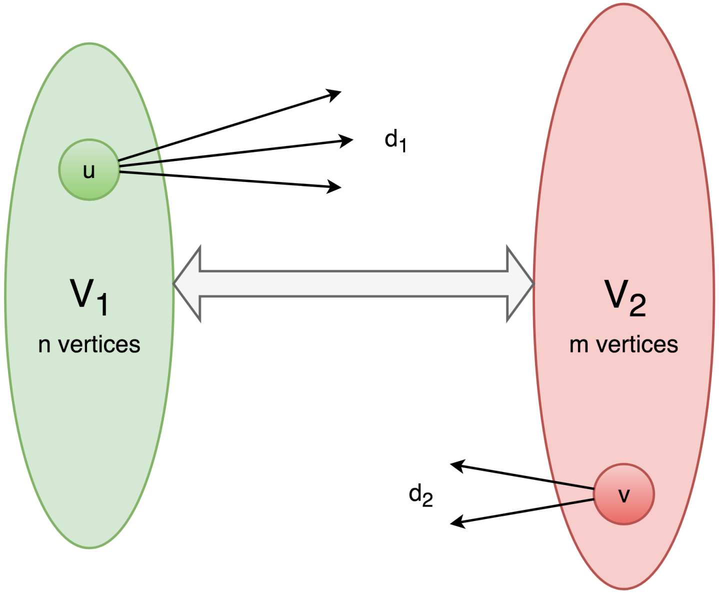
Expander Mixing Lemma, bipartite version:

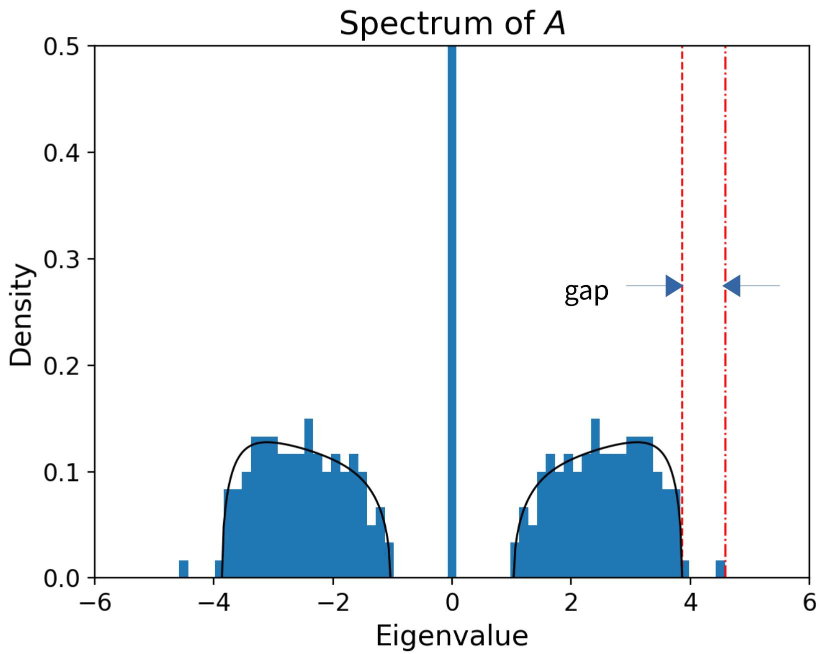
$$\left|\frac{E(A,B)}{|E|}-\frac{|A||B|}{nm}\right|\leq\boxed{\frac{\lambda_2}{\sqrt{d_1d_2}}}\sqrt{\frac{|A||B||A^c||B^c|}{(nm)^2}}$$

Gap

A proof in: De Winter, Schillewaert, Verstraete (2012)

Random graphs are good expanders





Gerandy Brito
GA Tech Computing

Ioana Dumitriu
UCSD Math

Theorem:

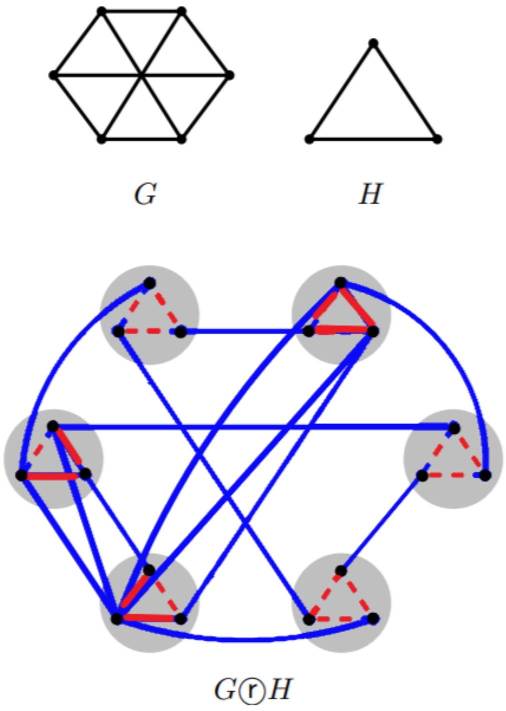
$$\lambda_2 \leq \sqrt{d_1 - 1} + \sqrt{d_2 - 1} + \epsilon_n$$

Study non-backtracking matrix

Friedman (2003)
Alon (1986)
Brito, Dumitriu, Harris in press

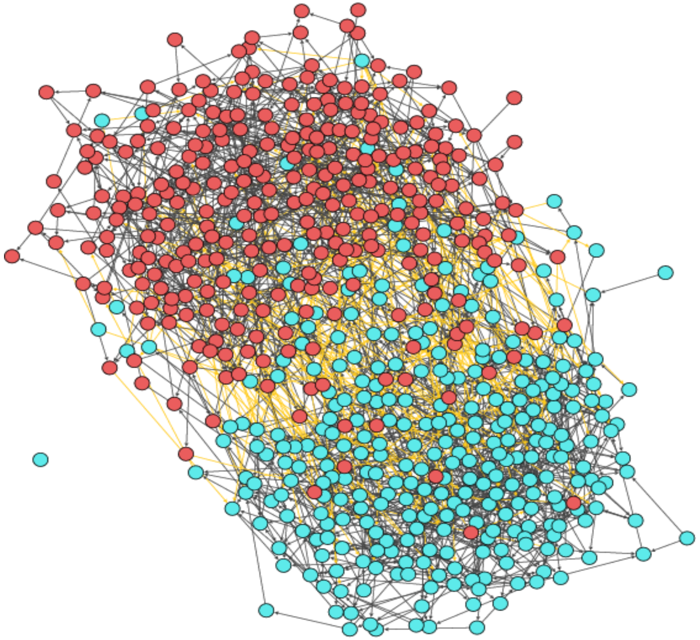
Deterministic expanders

- Algebraic constructions
- Zig-zag product
  - Reingold, Vadhan, Wigderson (2000)
- “Derandomization” big idea in theoretical CS



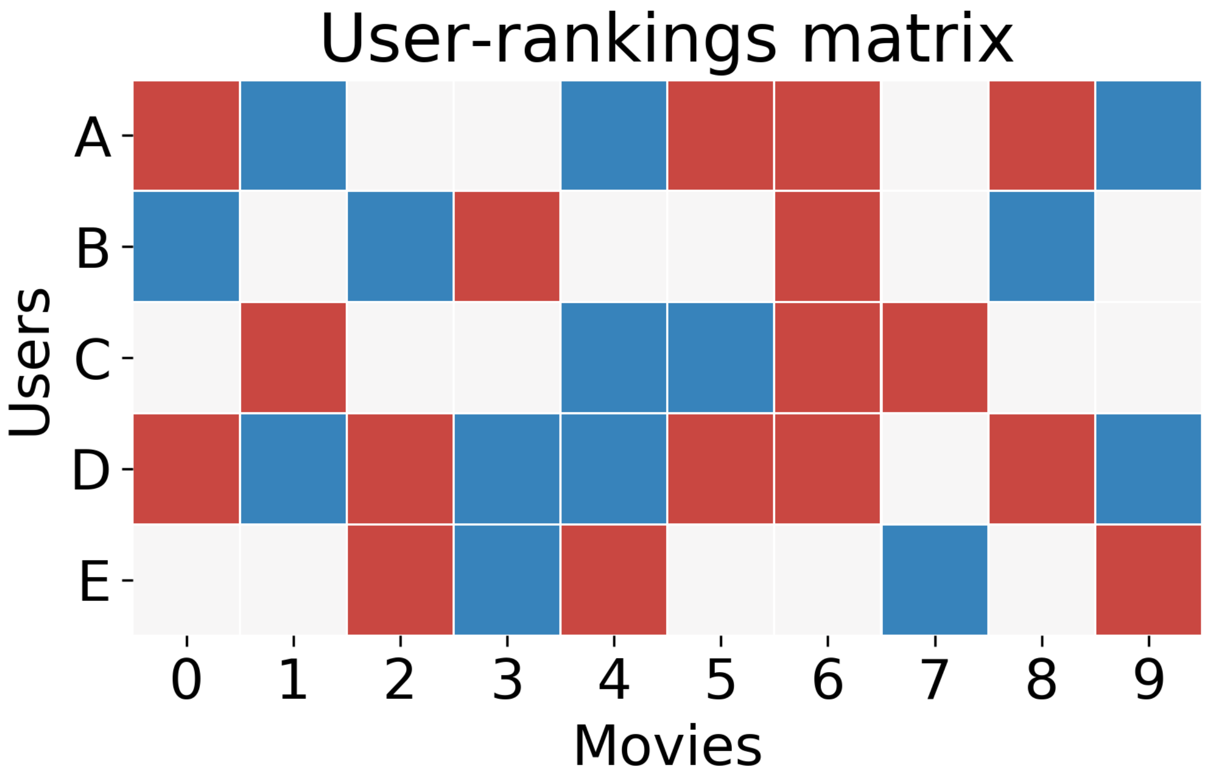
Applications of expander graphs

- Mixing rates of Markov chains
- Dynamics on networks, e.g. synchronization
- Community detection / spectral clustering
- Error correcting codes
- **Matrix & tensor completion**



Tensor completion

The NETFLIX problem

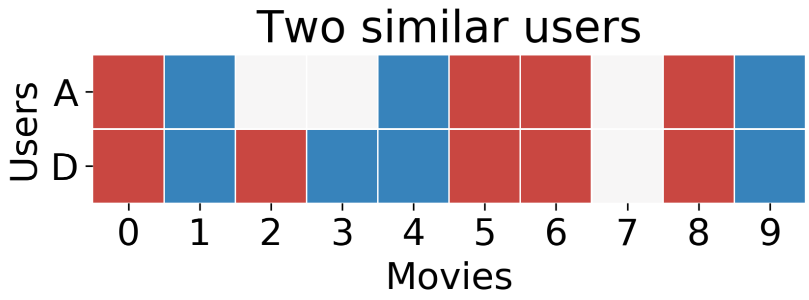


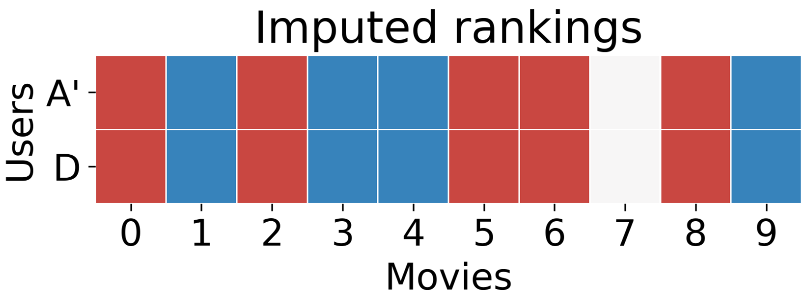
https://krishnaswamylab.github.io/tutorial/imputation_and_netflix/

The NETFLIX problem

Matrix completion

Rows are repeated
=
Low rank



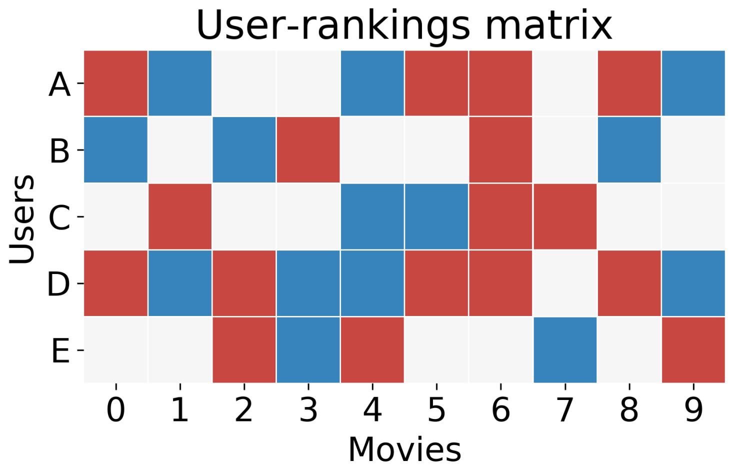


Proxies for rank:

- Sum of singular values
- Norms of factor matrices

https://krishnaswamylab.github.io/tutorial/imputation_and_netflix/

Connecting back to networks

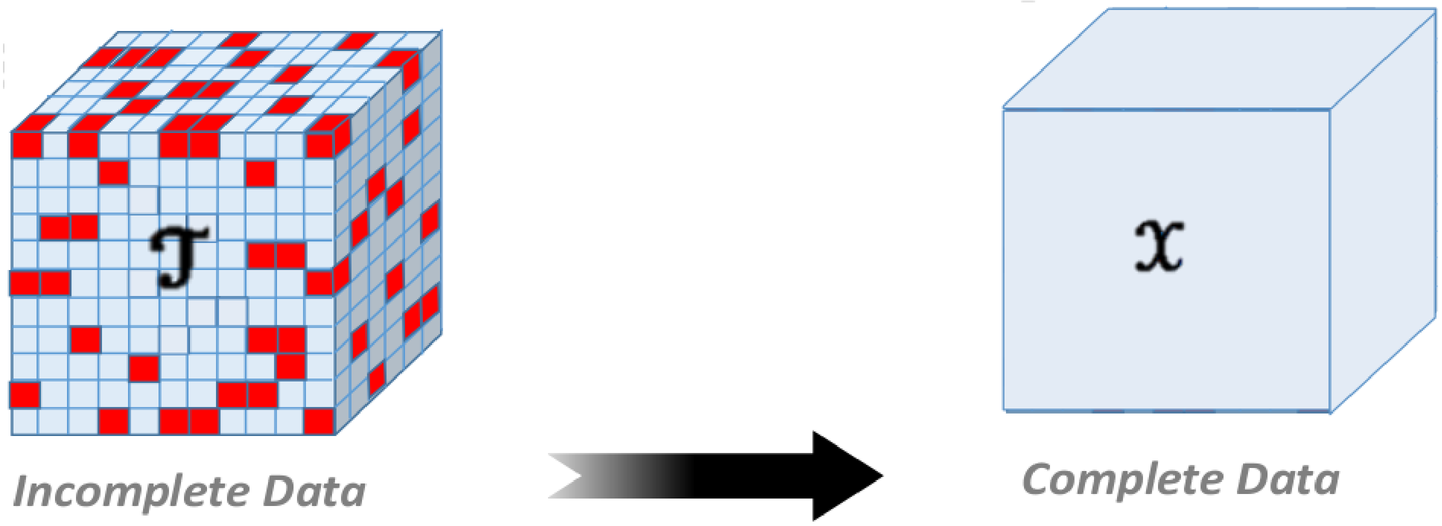


Data points are edges in a graph

$$(i, j) \in E \iff \text{entry } (i, j) \text{ is observed}$$

Tensor completion

- Use low-rank structure to infer missing data



Song, Ge, Caverlee, Hu (2019)

Tensor completion

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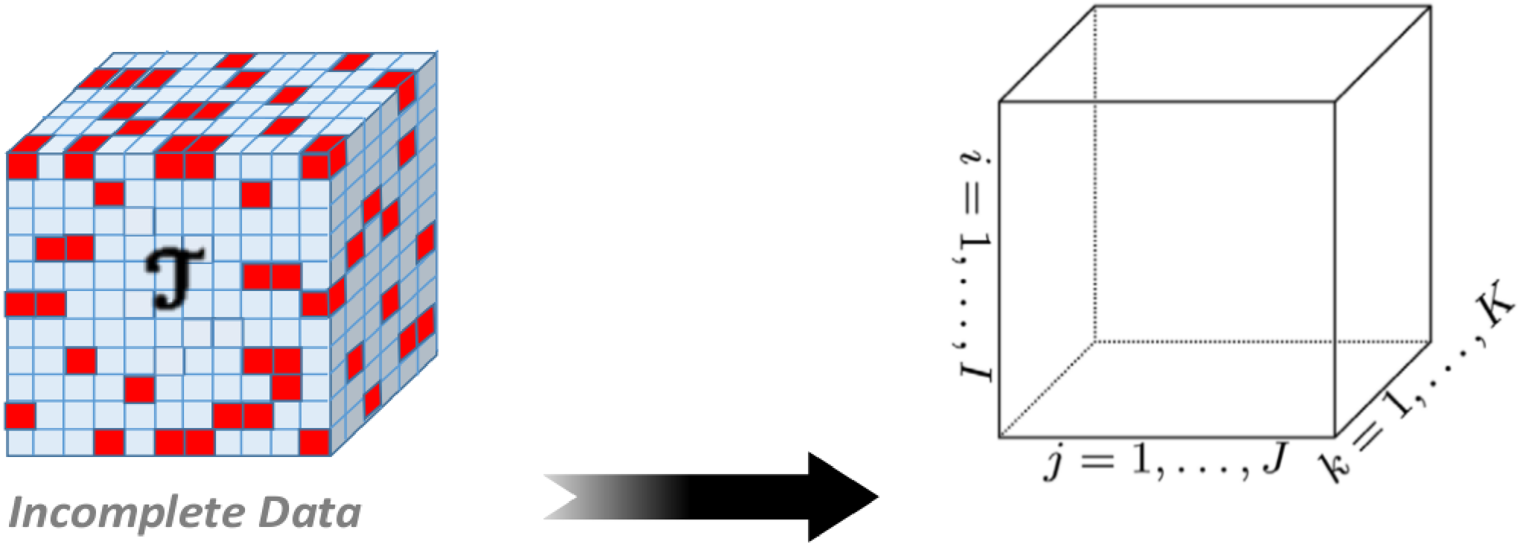

Excellent introduction:
Kolda & Bader. SIAM Rev (2009)

A third-order tensor: $\mathcal{X} \in \mathbb{R}^{I \times J \times K}$

Song, Ge, Caverlee, Hu (2019)

Tensor completion

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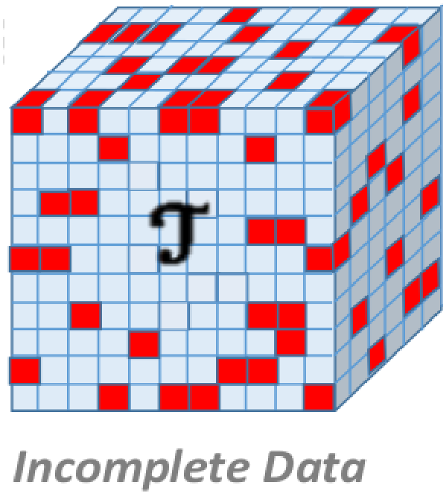

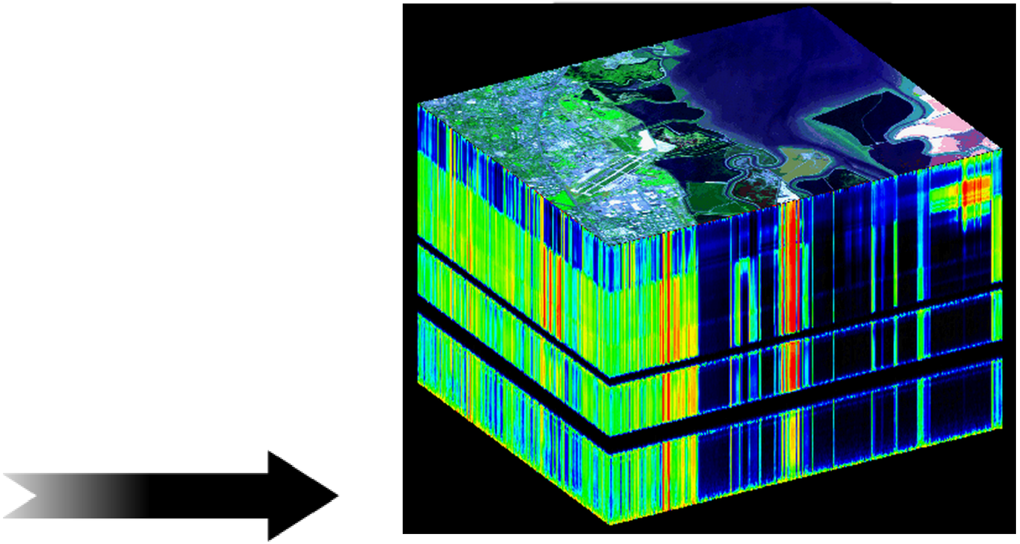

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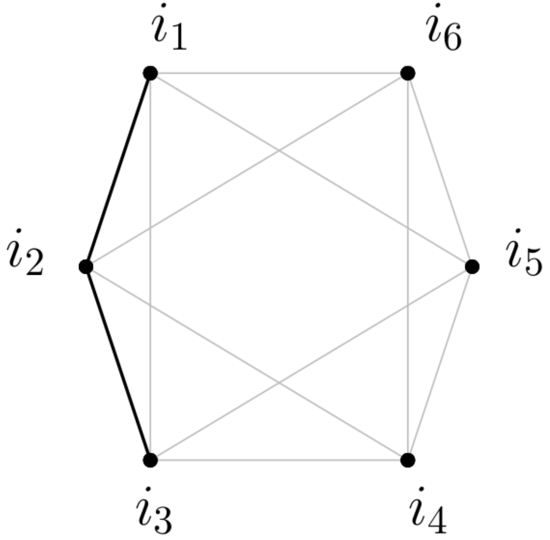
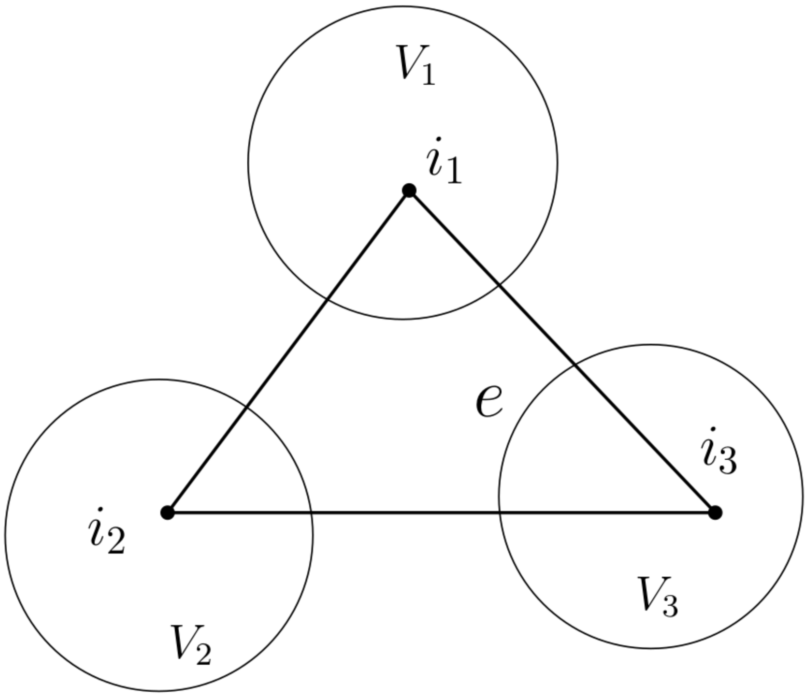
Hyperspectral imaging
NASA/JPL-Caltech

Song, Ge, Caverlee, Hu (2019)

Hypergraph observations



G expander



nd^{t-1} hyperedge
"observations"

H hypergraph
sparse adjacency tensor

Alon et al (1995)
Bilu & Hoory (2004)

Max-quasinorm of a tensor

$$\|T\|_{\max} = \min_{T=U^{(1)}\circ\cdots\circ U^{(t)}} \prod_{i=1}^{t} \|U^{(i)}\|_{2,\infty}$$

- Bounds nuclear norm of sign tensors via Grothendieck's inequality
- Depends on *r* and *t* but **not** *n*

Ghadermarzy, Plan, Yilmaz (2018)

https://creativeeducator.tech4learning.com/v07/articles/Field_of_View_v7

Expansion $\implies$ completion

Suppose we solve

$$\hat{T} = \mathrm{argmin}_{T'} \|T'\|_{\max} \text{ s.t. } \|\Omega * (T' - Z)\|_F \leq \delta$$

Related work:
Ghadermarzy, Plan, Yilmaz (2018)
Heiman, Schechtman, Shraibman (2014)
Brito, Dumitriu, Harris (in press)

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observation mask
i.e. adjacency tensor

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error tolerance

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data

Theorem: $$\frac{1}{n^t}\|\hat{T} - T\|_F^2 \leq C_t \|T\|_{\max}^2 \frac{\lambda}{d} + 4\delta^2$$

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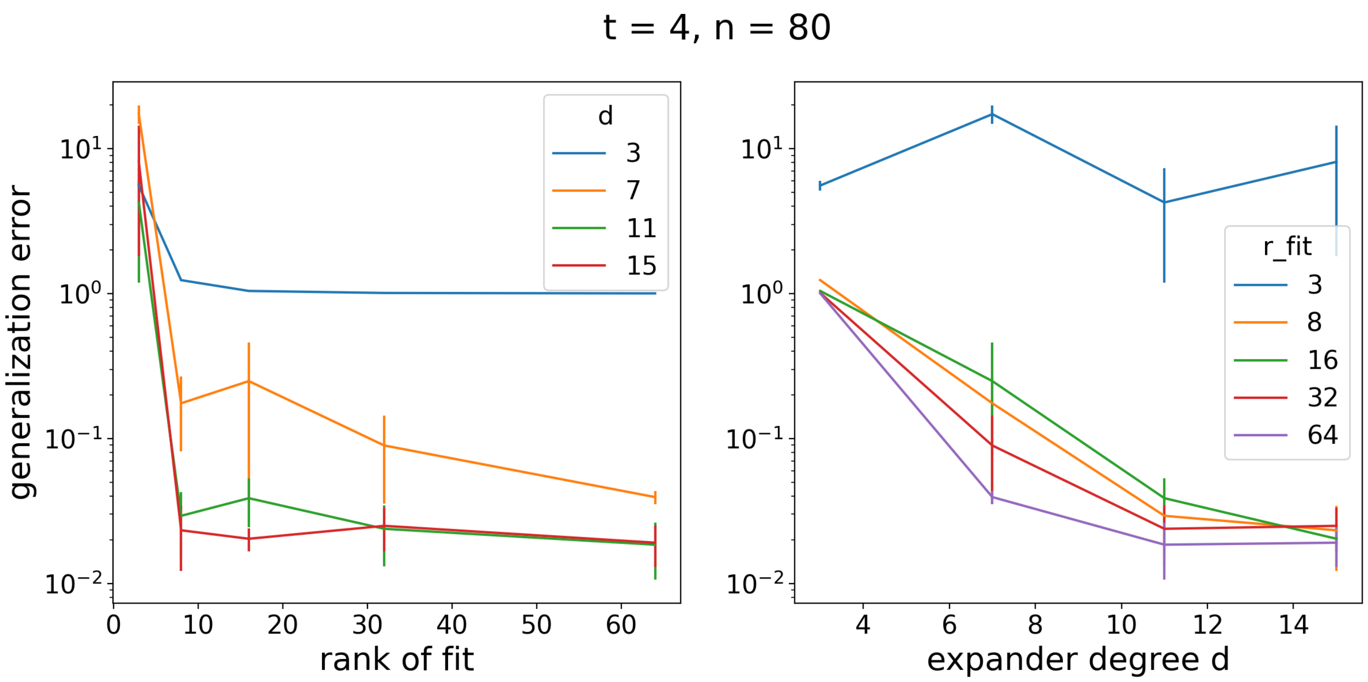
Linear sample complexity!

$$|E| = O\left(\frac{\|T\|_{\max}^{4t-4}}{\varepsilon^{2(t-1)}} \cdot n\right)$$

Practical algorithm works well



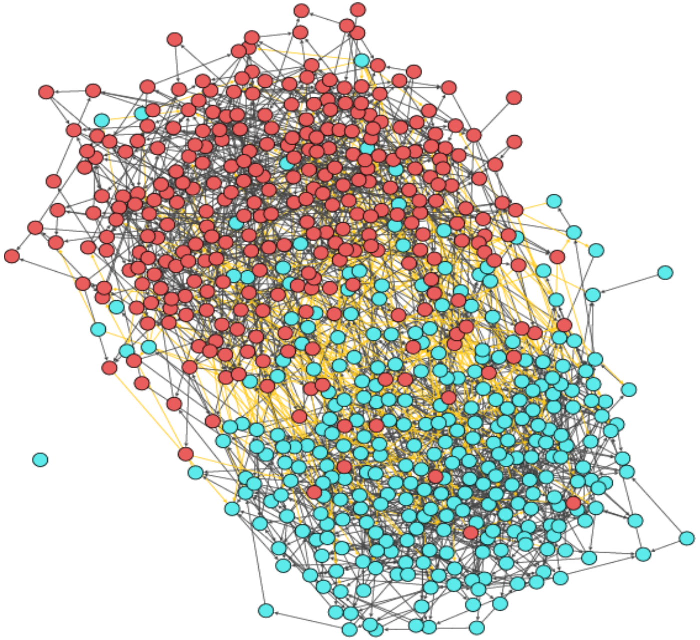
0.04% observed

https://github.com/kharris/max-qnorm-tensor-completion

Our results

- Improved understanding of the tensor *max-quasinorm*
  - Rank bounds, relationship other norms
- Hypergraph sampling model
  - Construction from expander graphs, new mixing inequality
- Deterministic bound of generalization error for completion
  - Linear sample complexity
- Numerical method which works well even with few samples

Harris, Zhu (in revisions)



Joshua Mendoza, Liu et al (2012)

Thank you for listening

and mind the gap

Scott Rinckenberger

Acknowlegements

Deterministic tensor completion with hypergraph expanders

In revisions

Yizhe Zhu

Spectral gap in bipartite biregular random graphs with applications

Comb Prob Comp, in press

Gerandy Brito

Ioana Dumitriu

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Papers & more:
https://glomerul.us